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INTRODUCING 'REGION GEOMETRY' IN MATHEMATICS

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ABSTRACT

In this work, a generalization of the existing classical subject Geometry is done by, introducing a new geometry called by "Region Geometry". It is observed that the classical subject Geometry is R-Geometry, a special case of the subject "Region Geometry" corresponding to the trivial region R. Every region A generates its own region geometry called by 'A-Region Geometry' subject to fulfilment of few conditions. Although this is a purely theoretical work, but this theory is expected to cater to Mathematics and all branches of STEM including AI, Data Science, modern Statistics etc in a new direction.

Keywords: region, 1-D complete region, region geometry, region coordinate plane, object coordinates, object point, object line, object circle.

INTRODUCTION

In this work, a new kind of geometry called by "Region Geometry" is introduced, and it is established that the classical Geometry is a special case of Region Geometry. It is the beginning of the notion of Region Geometry, which is at a baby stage in this work but is expected to grow very fast because of the requirement of it in the present era of AI in this century. History of the evolution of Geometry is very interesting. A large number of books, journal publications are available in the literature. The work here is mainly a work of school level. The books (Loney, 1875, 1923) are very popular to the school students of almost all the countries of the world for learning and practicing school Geometry. In school geometry, we teach the students about number line, coordinate plane, x-axis, y-axis, quadrants, x-coordinate of a point, y-coordinate of a point, location of a point on the coordinate plane, line segment, ray, line, etc. These basic and fundamental concepts are generalized in Region Geometry. In the work (Biswas, 2026a), a very important and useful algebraic structure 'region' is introduced to fix few serious hidden gaps in the giant subject Algebra. The works (Biswas, 2026a, 2026b) are the pre-requirements to introduce this new and generalized kind of
Geometry "Region Geometry".

It is justified in length in (Biswas, 2026a) with rigorous amount of analysis that most of the elementary, important and frequently practiced algebraic computations can NOT be validated by the existing giant ALGEBRA, although being practiced during past centuries. This statement seems to be an absurd or unimaginable statement, but it is true. For details one could see (Biswas, 2026a).

However, some portions of the works (Biswas, 2026a, 2026b) are reproduced in the next section here as the ready reference and for quick visit by the readers.

It is observed in (Biswas, 2026a) that the existing subject "Algebra" is having few serious gaps paralyzing it to support many types of mathematical computations (including identities, squares, cubes, nth power of an algebraic expression, CrossMultiplication rule, etc to name a few only out of many) being frequently done and practiced by us in all areas : Mathematics, Physics, Statistics, Chemistry, Space Science, Cosmology, ….etc. All the scientists till today, mathematicians, physicists, chemists, statisticians, cosmologists, space scientists, economists, …, are lucky that they got results without facing any contradiction or without facing any computational error. It is very surprising statements and initially at this moment could seem to be statements of **unimaginable matters**. But it is true. **The article (Biswas, 2026a) explains rigorously how this magic happened so far in research works of world scientists.** The work is done to introduce a new algebraic structure Region with a very genuine reason (justified in length in (Biswas, 2026a)), but is not done to extend the existing volume of literature of the subject "Algebra" **unnecessarily**. <u>An algebraist can</u> <u>define an infinite number of new algebraic structures.</u> The objective of the work(Biswas, 2026a) is not just for the sake of discovering a new algebraic structure, but to recognize and identify a major gap of the giant subject 'Algebra' lying hidden so far in the existing vast literature of it, and then to fix the gap.

Consider a list of key algebraic structures in Algebra which are **:** Semi Group, Group, Ring, Integral Domain, Module, Field, Linear Space, Algebra over a field and Extensions, Associative algebra over a field, Division Algebras, Cayley–Dickson algebra, Clifford Algebra, universal algebra, etc. For the purpose of frequent uses of these important algebraic structures, let us call this list by the name **"Key List'.**

The work(Biswas, 2026a) explains in length how this magic happened so far in research works of world scientists during last centuries, without producing wrong results. Consider few examples :-

Example 2.1

Consider a very simple instance from elementary algebra, a type which is very frequently used by the secondary school students and ofcourse by the world researchers, is the equality (identity) **I** of following type:

□ □□ □*ba* □□ □□ □□ □*xy* = □□□____*ba x*□□*y* □□□ ,
□ □

where x, y are in an algebraic structure A (assume that the name of the algebraic structure A is not known at this moment) and a, b are two members (scalars) of some known field F.

We would like to say that this beautiful and simple identity is not valid i.e. **'can not be verified'** in any algebraic structure mentioned in the **Key List** (i.e. **'can not be verified'** in Group, in Ring, Integral Domain, Field, Linear Space, F-Algebra, Associate Algebra, Division Algebra, Normed Division Algebra, Cliffored Algebra, Cayley Algebra, Cayley–Dickson algebra or in any existing standard brand of algebraic structure alone in general), **by virtue of their respective definitions and independently owned properties**.

The above issue has been fact since last centuries.

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See the justification presented below, about the above identity **I**.

Justification

It is because of the reason that **:**

(1) since division operations are involved in both LHS and RHS of the above expressions **I**, the unknown algebraic structure A can not be a 'F-algebra' by virtue of its definition and independently owned properties.

(2) next, if it is not a 'F-algebra' then let us check whether it could be a division algebra **D** (remembering that every field is also a division algebra).
Then the following are fact by virtue of the definition and independently owned properties of division algebra **D**:-

(i) the LHS expression □ □□ □□ □$_b{}^a$ □□ □□ □□ □$^x{}_y$ of **I** can be well written to be equal to the expression □□□a.—$_b{}^1$ □□□ □ (x^*y^{-1}) in the division algebra **D**, (ii) but next the expression □□a.—$_b{}^1$ □□□ □ (x^*y^{-1}) **can't be written** to be equal to

□

the expression (a□ x) *□□ —$_b{}^1$ □ y□1□□□ in the division algebra **D**, by virtue of the

□

definition and own properties of "division algebra".

[although, it is true that in the division algebra **D,** by virtue of its definition and owned properties, (a□ x) *□□□ —b1 □ y□1□□□ can be written equal to the expression □□□——$ba\ x$□□y □□□].

Consequently, in a division algebra **D,** the expression □ □□ □□ □$_b{}^a$ □□ □□ □□ □$^x{}_y$ can **not** be accepted

in general to be equal to □□□——$b^a\ x$□□y □□□, by virtue of its definition and own properties. This is

a great failure of Division Algebra, whereas the identity(equality) of type **I** are very frequently being used by the mathematicians.

Out of all the existing recognized algebras of the **Key List**, none can prove the above identity/equality !! This is a serious setback of the existing volume of literature of the subject 'Algebra'. And this simple example shows a serious gap in the subject "Algebra" !!

Fortunately the set R of real numbers satisfies **some additional interesting properties**, remaining unknown so far, beyond the properties of it possessed by virtue of the definition and properties of the algebraic structure **division algebra**. And that is the hidden reason why the mathematics has not been facing any problem and have not been getting any incorrect final results or contradictory results in R, while applying the above type of identities **I** in mathematics exercises and computations in science subjects or engineering subjects or in any STEAM subject.

Example 2.2

With exactly the same argument as in Example 2.1 above, it can be observed that if an algebraic equality (in fact it is an identity) **J** of type given by

2
□□□($a\ x$□)□ ——$b\ y$□1 □□ □~~$a\ x2$~~□ 2 □—1 □□2a□□□ □x
□ $b\ y2$ □ 2□□ b □□ □ □□ □y

happens to be a valid identity (i.e. can be computed and verified) in an algebraic structure A where x, y □A, a and b being members (scalars) of some given field F, then it can be observed that **each** of the following five statements are true:-

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a. A is not a group, or a ring, or integral domain or module, field, linear space, etc by virtue of their respective definitions and independently owned properties.
b. A is not an 'algebra over a field F' (F-algebra) by virtue of its definition and independently owned properties.
c. A is not an 'associative algebra over a field' by virtue of its definition and independently owned properties.
d. A is not a 'Division Algebra' by virtue of its definition and independently owned properties.
e. A is not any standard existing brand of algebraic structure or of any standard extension, by virtue of its definition and independently owned properties.

Undoubtedly, if the above identity named by **J** (and also the identities of Example 2.1) can not be validated by any existing algebraic structure A or existing extensions, then our giant subject "Algebra" must identify: **"Who is this unknown algebraic structure A in whom these identities are valid by virtue of their respective definition and independently owned properties?"** Otherwise the subject Algebra will remain as a subject having few serious gaps.

Not only for square of an algebraic expression, the same is true for cubes, for nth power etc.

Consider few more examples below:-

Example 2.3 Cross-Multiplication Rule is Invalid ?

The following type of simple computation of **'Cross-Multiplication'** C of elementary algebra like :

if $\frac{a x}{b y} = \frac{c z}{d t}$ then $(ad)(x \cdot t) = (bc)(y \cdot z)$ (and conversely),

where x, y, z, t are in an algebraic structure **A** (the name of this algebraic structure **A** is not known at this moment) and a, b, c, d are the members (scalars) of some given known field F, **'can not be verified'** (i.e. can not be validated) in general in any algebraic structure of the **Key List**, just by virtue of their respective definitions and independently owned properties. The justification in brief regarding this very popular

'Cross-Multiplication' property C is presented below :-

Justification:

The justification is in fact similar to what made in the case of Example 2.1 above.

(1) since division operations are involved in both LHS and RHS expressions, **A** can not be a 'F-algebra' by virtue of its definition and independently owned properties.
(2) in the next step, if it is not a 'F-algebra' then let us check whether it could be a division algebra **D** (remembering that every field is also a division algebra). Then the following are fact by virtue of the definition and independently owned properties of division algebra:-

Suppose that the given equality $\frac{a x}{b y} = \frac{c z}{d t}$ being considered here is in the algebraic

$b\,y \quad d\,t$

structure **A** which is a "division algebra"**.**

Then, we have from this equality that the equality $(a\,x)(b\,y)^{-1} = (c\,z)(d\,t)^{-1}$ is also valid in the division algebra **A.** Upto this step-point, there is no issue. But after this step, the mathematics is blocked and can not proceed further for any next step in the division algebra **A**. Because this can not yield the next step of the computation expected to be as below $(ad)\,x\,t = (bc)\,y\,z$ in the division algebra **A** just by virtue of its definition and its independently owned properties of the algebraic structure 'division algebra'.

Example 2.4

It can also be seen that this type of simple square identity **I** like :

$$\left(\frac{a\,x}{b\,y}\right)^2 = \frac{c\,x^2}{d\,y^2} \quad (\text{where } c = a^2 \text{ and } d = b^2)$$

(where x, y, z, t are in an algebraic structure A (the name not known at this moment, what is the name of this algebraic structure) and a, b, c, d are members (scalars) of some field F) **'can not be verified'** (i.e. can not be validated) in any algebraic structure of the **Key List**, by virtue of their respective definitions and independently owned properties.

Then, the immediate questions that arose in our mind are :-

- "What is the minimal algebraic structure which can validate the above examples?" Or
- "What could be the minimal algebraic structure in which the above type of identities like **I** or the above type of cross multiplication results like C, etc are valid?". Or
- "In what minimal algebraic structure, the above type of identities like **I** or the
  above type of cross multiplication results like C, etc can be verified?".

Note carefully that, in the above four examples (and many such type of similar examples) we are talking about important algebraic computations like : **identities**, **square or cubes or nth power of an algebraic expression**, **cross-multiplication**, etc which are very frequently used and very importantly used for computations by the students and researchers in mathematics, statistics, science subjects, engineering subjects, in any STEAM subject.

An algebraist can not answer what is this unknown algebraic structure A in the above four examples. He must answer a unique name of an algebraic structure, because of the reason that these type of examples are of tremendous practices by the students and mathematicians in their daily computational works and activities. Is it a Group?, or Ring, Integral Domain, Field, Linear Space, F-Algebra, Associate Algebra, Division Algebra, Normed Division Algebra, Cliffored Algebra, Cayley Algebra, Cayley–Dickson algebra or any standard algebraic structure (assuming that division by zero element is not allowed), etc??. For a possible answer, because of non-availability from the existing huge volume of algebraic structures of the subject 'Algebra', the algebraist has to think of a permutation/combination of the various existing algebraic structures (and their extensions) to discover a possible result. But, he might seek to make a unique identity for this algebraic structure A (a minimal algebraic structure

A) to define it in an independent and atomic way, as the algebraists did the same earlier to define ring, field, division algebra, ... etc instead of doing a permutation/combination; because
'Algebra' must be sound and complete in this sense.

Consequently there a genuine need to identify this minimal algebraic structure, which is hidden so far, un-discovered far, for practicing the giant subjects elementary algebra and higher algebra more appropriately. And thus a new algebraic structure called by **"Region"** took birth in (Biswas, 2026a) which is the minimal algebraic structure which can validate the frequently practiced algebraic computations of school mathematics and of higher levels. The enormous and very unique potential of Region Algebra lies upon the fact that "this is the **minimal** algebra" by virtue of the definition and independently owned properties, which can give full license to the mathematicians to practice all the existing simple and frequently useful results, equalities, identities, formulas etc. of elementary algebra, unlike the capabilities of any of the existing algebraic structures of the subject 'Algebra'. In other words, region is the **minimal** algebra which can authenticate the elementary algebra. **None of the existing algebraic structures as listed in the 'Key List' has this ability.** And hence region is the most important and most powerful algebraic structure in Algebra.

The beautiful properties and results fulfilled by the set R of real numbers are being so far fluently used by the mathematicians <u>assuming R to be a division algebra</u>, but without knowing that they are actually using the 'region' properties of R fluently, not the properties of division algebra only. It is like a car driver on road without valid (authorized) driving-licence has been driving always error-free just because of the reality that there is no accident happening, no collision recorded, so far. Consequently, region is not only the **minimal** algebraic structure to remove few serious gaps in the subject 'Algebra', but also the most useful and most powerful algebraic structure in the world of computations.

It is very important to see the word **minimal** used here, as explained in (Biswas, 2026a). We call this new algebraic structure by **"Region".**

Region

Consider a non-null set A with three binary operators □, * and □ defined over it such that for a given field (F, +, .), the following fourteen(14) conditions are satisfied:-

A, □) is an abelian Group with respect to addition:

(1) if x, y, z □A then (x□y)□z = x□(y□z).
(2) □ 0_A □A such that x□0_A = x = 0_A □x, □x□A.
(3) if x□A, □ y□A such that x□y = 0_A = y□x.
(4) □x, y □A, x□y = y□x.

(A, *) forms an Abelian Group (excluding the element 0_A) with respect to multiplication:

(5) if x, y, z □A then (x□y)□z = x□(y□z).
(6) □ 1_A □A such that x□1_A = x = 1_A □x, □x□A.
(7) if x□A, □ y□A such that x□y = 1_A = y□x.
(8) □x, y □A, x□y = y□x. **Distributive Properties :**
(9) □ x, y, z □A then
  (i) x□(y□z) = (x□y)□(x□z).
  (ii) (x□y)□z = (x□z)□(y□z).

Scalar Multiplication :

(10) Scalar multiplication of x ∈ A by element k ∈ F, denoted by k□ x is to be in A,

(11) k□(m□x) = (k.m)□ x, where x ∈ A, and k, m ∈ F.

(12) k□ (x □ y) = k□x □ k□ y, where x, y ∈ A, and k, m ∈ F.

(13) (k + m) □x = k□x □m□ x, where x ∈ A, and k, m ∈ F. **Compatibility with the scalars of the field F :**

(14) A satisfies the property of "Compatibility with the scalars of the field F",
i.e. (k□x) * (m□ y) = (k.m)□ (x*y)
□k, m □ F and □x, y □ A.

Then the algebraic structure (A, □, *, □) is called a **Region** over the field (F, +, **.**).
If there is no confusion, we may simply use the notation A to represent the region (A, □, *, □), for brevity. It may be observed that if (A, □, *, □) be a Region over the field (F, +, **.**), then (A, □, *) is a field and (A, □, □) is a linear space over the field (F, +, **.**). However, being a field as well as a linear space does not suffice to become a region.
The above definition of region may be presented alternatively as below:-
Consider a non-null set A with three binary operators □, * and □ defined over it such that for a given field (F, +, **.**), the following three(3) conditions are satisfied:- (i) (A, □, *) forms a field,

(ii) (A, □, □) forms a linear space over the field (F, +, **.**), and

(iii) A satisfies the property of "Compatibility with the scalars of the field F",
i.e. (k□x) * (m□ y) = (k**.**m)□ (x*y) □k, m □ F and □x, y □ A.

Clearly a region is not just a division algebra only, but having a lot of amounts of more mathematics in it. A division algebra, in general, is not a region. However, one could try to view a region by permutation/combination of some of the existing classical algebraic structures in other ways. For instance, the following two theorems follow directly from the axioms (definition).

Theorem 2.1

Let (A, ⊕, *, ·) be a Region over a field F. Then the following are true:

1. (A, ⊕, *) is a commutative field,
2. (A, ⊕, ·) is a vector space over F,
3. (k·x)*(m·y) = (km) ·(x*y).

Theorem 2.2

Every Region is a commutative division F-algebra.
Consider the following interesting simple problem on Algebraic Computations.

Problem 2.1.

Obtain an expression for x in terms of y and t from the following equation in the real region (A,□,□,□) :

3□ x* y = 2□ y□3□ t , where
x , y (□ $0_{\mathbf{A}}$), t □ **A**.

Solution : We have the following equation in the region **A :**

3□ x * y = 2□ y□3□ t

Using the properties of region, we then can write

$\frac{1}{3}$□ (3□ x*y) = $\frac{1}{3}$□ (2□ y □ 3□ t)

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or, (x*y)*y -1 = □□ _23 □ □y t□□□*y -1 □

or, x * (y * y -1) = □□ _23 □(y y□ □1)□□□ □(t * y -1) □

or, x*1A = □□ _23 □1A □□□ □(t * y -1) □

or, x = □□ _2 □ □1A _t □□, which is the solution.

or, x*y = $\frac{2}{3}$□ y□1F □ t

or, x*y = $\frac{2}{3}$□ y□t

□3 y □

or, □□□_ 13□3□□□ □ (x*y) = □□□ _13 □(2□ y)□□□ □ □□□ _13 □(3□t)□□□

or, 1F □ (x*y) = □□ _13 □2□□□ □ y □ □□□_13□3□□□ □ t

□

An Interesting Analysis about the above solution

Let us analyze now the solution to the above Problem 2.1. For this, we begin our analysis with an element of imagination with the following two points:-

Point (i) : It is mentioned in the problem statement that the algebraic structure **A** is a region. But, let us imagine a situation that the identity of the algebraic structure **A** in the above Problem 2.1 is "unknown" to us at this moment, and also let us accept that ***Point (ii) :*** let us also accept that the solution steps presented above are well valid and correct in this "unknown" algebraic structure **A.**

Now, in the above solution steps we see that :-

There are few steps which are allowed by virtue of the definition and properties of 'vector space', and there are few steps which are allowed by virtue of the definition and properties of 'division algebra'. It is obvious that a 'division algebra' can not give license to all the steps of the above mentioned solution-method by virtue of its own definition and independently owned properties (for example, 'compatibility with scalars' is not a licensed step in division algebra, even not the commutative property). Besides that, see that division operations are executed in the solution steps. Hence **A** can neither be just an 'algebra over a field' nor an 'associative algebra over a field'. *Consequently, considering the validity of "all the involved operations collectively" in the steps of the above mentioned solution-method, it is now obvious that this unknown algebraic structure A of this Problem 2.1 has to be at minimum a 'region', not less (i.e. not a Division Algebra, not any of the existing standard algebraic structures listed in the* ***Key List****). Otherwise, the above problem can not be solved for x in the algebraic structure A, showing a gap in the subject "ALGEBRA". There must be a unique identity of A, because the existing subject Algebra (with all its existing algebraic structures and extensions) fails by its existing vast rich literature to solve the above Problem 2.1 !!! And this unique minimal algebra A is "Region".*

Let R be the set of real numbers, '+' be the ordinary addition operator in R and '.' be the ordinary multiplication operator in R. Consider the field (R,+, .) of real numbers, and the linear space (R,+, .) over the field (R,+, .) . Then the algebraic system (R,+, . , .) forms a region over the outer field (R,+ , .). This region (R,+ , . , .) plays a very important role in our daily life computations, in particular in school level elementary algebra. The content of the syllabus and corresponding instructions at

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school level algebra is based on the platform of this region (R,+, . , .), not on the platform of any standard algebraic structure like groups, rings, integral domains, fields, linear spaces, algebra over a field, associative algebra over a field, division algebra or any existing algebraic structure. Let us name this region (R,+, . , .) in short by the word
"RR". The region RR is the most useful region in all the branches of Mathematics, Statistics, Science, Engineering etc i.e. in all STEM subjects.
The interesting properties of the region RR are that:
(i) its inner field is (R,+, .),
(ii) its outer field is also (R,+, .),
(iii) all the three multiplication operators are same, and (iv) all the three addition operators are same.
A region (A, □, *,□) over the field (F, +, **.**), is called a Real Region if its outer field F is the classical field R of real numbers. The regions RR, C are examples of real region. In Region Algebra, the **characteristic** of a region A denoted char(A) is defined to be the smallest number of times one must use its multiplicative identity 1_A in a sum to get the additive identity element 0_A. A region is said to have characteristic zero if this sum never reaches the additive identity. For example, for the region RR we have Char(RR) = 0.
In our work here, we use the following notations fluently:
R = set of all real numbers, R^+ = set of all positive real numbers, R^- = set of all negative real numbers, $R^{\geq 0}$ = set of all non-negative real numbers. As prerequirement for introducing "Region Geometry", few concepts are reproduced here from the work (Biswas, 2026a, 2026b),
Consider a real region A = (A,□,*,□). Suppose that A forms a chain with respect to a total order relation (say, denoted by the notation '□'). Then the real region A is called a **chain region** with respect to the total order relation '□'.
A real region A = (A,□,*,□) is called a **Partitioned Region** if the following conditions are satisfied :

(i) A is an infinite region,
(ii) A is a chain region with respect to a total order relation '□', and (iii) the characteristic of A is zero.

Here A is called a 'partitioned region' because of the fact that it induces a partition P_A of A into three mutually disjoint non-null sets denoted by A^+, A^- and $\{0_A\}$ such that (i) $A^+ = \{ a : a$ □ A and $0_A < a\}$
(ii) $A^- = \{ a : a$ □ A and $a < 0_A\}$.
Clearly, □a □ A^+, ~a □ A^- and □b □ A^-, ~b □ A^+.
(**Note** : It may be recalled from the properties of the chain that : $a < b$ iff $a \leq b$ and $a \neq b$, where "≤" is the total order relation of the chain A, and similarly $a > b$ iff $b \leq a$ and $b \neq a$).
This partition P_A, once made, is regarded as an absolute partition of the region A corresponding to its total order relation '□' in the sense that this partition generates the sign of every object of the complete region A, positive or negative, which will remain absolute throughout the complete literature henceforth . However for a different type of total order relation defined over the region A we will get a different partition of A. But the set $\{0_A\}$ is common to all such possible partitions.

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Consider an infinite region A = (A,□,*,□). The **extended region** of the region A is the

'region itself with all its infinity objects, if any'. The infinity objects are not basically the core member of the region A. For the RR region and the C region, the concept of infinity is known to us. Infinity is not an element of the set R of real numbers. Instead, the real line extends infinitely in both the directions, leading to positive infinity (+□) and negative infinity (-□), which are included in the ***extended*** **real number system** but not in R. An analogous concept gives the notion of Complex infinity. It is like a complex number with infinite magnitude and an undefined argument. The most common representation is Riemann Sphere where all infinite paths are wrapped around a sphere. At this point of time we do not consider any method about 'how to find out all the infinity objects of an infinite region'. For our work here, we do not need to study all the existing advanced theories of infinities so far developed in Mathematics. However for a partitioned region the method is rather easier about considering the concept of infinity as mentioned below. And in this work all-through, whenever the notion of infinity is to be used, we must consider only those regions which are partitioned regions

Consider a partitioned region A = (A,□,*,□). If we now include two more objects $+□_A$ and $-□_A$ in the partitioned region A as two permanent guests, then the set A^E = A □{$+□_A$, $-□_A$} is the 'extended region' of the partitioned region A.

The two guest objects $+□_A$ and $-□_A$ are called infinities of the partitioned region A, and are defined as below: $x^{\underline{A}}$ where x_A ($\neq 0_A$) is any positive object of the partitioned region A, and (i) $+□_A$ =

0_A $z^{\underline{A}}$ where z_A ($\neq 0_A$) is any negative object of the partitioned

region A. (ii) $-□_A$ =

0_A

The extended region of the partitioned region A is denoted by the notation A^E. However, if there is no confusion then we may use the notation A itself to denote the extended region of A. Note that an extended region is not a region. For a partitioned region, it is just a superset of the set A containing two more objects.

An extended region A^E may also be called as 'extended real region' if the corresponding region A is a real region by virtue of the definition of a partitioned region. For the region C, there are many infinities to be included into it to call it an extended region. For example □a, b □ R, the object a+ib is an infinity object for C if either a or b or both are the infinity object of the region R. The extended region of C is denoted by the notation C^E. Further future study on the topic of extended region will make the literature richer. However, in our work here we use the case of the extended region of a partitioned region only.

We use the following notations in our work here :-

R = set of all real numbers, R^+ = set of all positive real numbers, R^- = set of all negative real numbers, $R^{\geq 0}$ = set of all non-negative real numbers.

Consider two non-null sets X and Y. A function f : X □ Y is said to be a '2-to-1 Bijective Mapping' if

(i) f is onto, and

(ii) □y □ Y, □ two and only two distinct (not same)

elements x_1 and x_2 in X such that
$f(x_1) = y = f(x_2)$.
For example, the function f : R-{0}□ R^+ given by $f(x) = x^2$ is a 2-to-1 Bijective Mapping. But the function g : R□ R^+ given by $g(x) = x^2$ is not a 2-to-1 Bijective Mapping.
Consider a set S. Surely we can not form a calculus over the set S unless we are aware of any algebraic structure which ensures us about the operations, axioms, properties defined over the set S. No calculus can be developed over an algebraic structure like Field, Division Algebra or any of the existing important algebraic structures by virtue of their respective definitions and independently owned properties. It must be necessarily to be a region at minimum. It is because of the reason that most of the results of elementary algebra are not valid in any of the algebraic structures listed in the **Key List** by virtue of its respective definition and independently owned properties. The minimal algebraic structure is region (see (Biswas, 2026a, 2026b) for details). For developing a new geometry, there must be a corresponding platform region over which the geometry will grow. But a natural question at this point arises: Can we develop a geometry over any given region S? Consider a partitioned region A = (A,□,*,□) with respect to the total order relation '□'. Then the region A forms a **Region Space** if the following conditions are satisfied :

(i) A is an extended region
(i.e. A is a region and two infinities are also included to it as permanent guests).
(ii) A is a normed complete metric space with respect to a norm $\|.\|$ and the corresponding induced metric $\rho(x, y) = \|x \sim y\|$, (i.e. $\|x\| = \rho(x, 0A)$).
(iii) The norm $\|.\|$ is a 2-to-1 bijective mapping from $A - \{0_A\}$ to R^+.

A real region which forms a region space is called a **"complete region".**
We will introduce later that by a complete region, we will always mean one-dimensional complete region (1-D complete region).
For instance, the region RR is a complete region with respect to the crisp order relation "Less Than or Equal To" denoted by the notation "≤" and the metric $\rho(x, y) = \| x - y\| = |x-y|$, where the norm is the classical norm defined over R.
The collection of all the complete regions is called the complete region universe Σ.
Consider a partitioned complete region A = (A,□,*,□). In earlier section, three partitions of a region is made which are denoted by A^+, A^- and $\{0_A\}$ where A is a chain region with respect to a total order relation '□'.
The elements of A^+ are said to be **positive objects** and the elements of A^- are said to be **negative objects**. The object 0_A is neither in A^+ nor in A^-, and so we say that 0_A is neither a positive object nor a negative object. The attribute of being positive or negative is called the sign of the object, and the object 0_A is not considered to have a sign of its own.
A line can be drawn on plain paper on which one point may be fixed to be the location for the zero object 0_A, and with all positive objects of A having their respective locations to the right and all negative objects of A having their respective locations to the left of the zero object 0_A. For a positive object x, the respective location on the line means the unique point on the right of 0_A at the distance $\rho(x, 0_A)$ from 0_A, and an analogous concept is true for a negative object x but on the left of 0_A. Thus the 'positive direction' of the line can be called to be XA-axis and the 'negative direction'

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of the line can be called to be XA1-axis. And the line which the objects of the complete region A is considered to lie upon is called the Object Linear Continuum Line for the complete region A (see Figure 3.1).

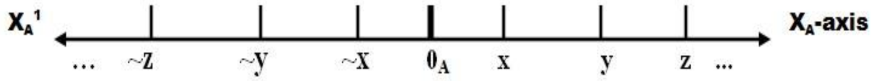

Figure 3.1 Object Linear Continuum Line of the complete region A, a general view

Thus, any point on the Object Linear Continuum Line of the complete region A is called an object point of A.

It is already mentioned earlier that, We use the following notations in our work here :

A^+ = set of all positive objects of the complete region A,

A- = set of all negative objects of the complete region A,

Let us use the following notation too:

$A^{\geq 0}$ = set of all non-negative objects of the complete region A.

For developing an Object Line corresponding to a region A, it must be necessary that A forms a region space. Consider the object linear continuum line and the corresponding X_A-axis. Since the region A is complete, there are no "points missing" from it (inside or at the boundary). Since A is a chain, every object of A has a unique address on this object linear continuum line and conversely i.e. corresponding to every address (point) on this 'object linear continuum line' (see Figure 3.1) there is a unique object of the region A. This property is called **"Completeness Property"** of this complete region A.

Consider a point x on the X-axis of the object linear continuum line corresponding to the region space A. Then for an infinitesimal small positive object Δx of the region A, the point (x□Δx) will be at a distance $\|\Delta x\|$ from the point x along the positive direction of X-axis and the point (x - Δx) will be at a distance $\|\Delta x\|$ from the point x along the negative direction of X-axis. By distance between two objects x and y lying upon the XX^1 Object Linear Continuum Line of the complete region A, we mean the corresponding metric distance ρ(x,y) of the normed complete metric space A.

By the distance between two objects x and y of the complete region A, we mean the corresponding metric distance ρ(x, y) of the normed complete metric space A. The distance of a positive object x_A from the origin is $\|x_A\| = \rho(x_A, 0_A) = x_a$, and the distance of a negative object $-x_A$ from the origin is = $-x_a$ (imposing minus sign). For example, see a collection of consecutive equi-spaced points on the object line as shown in the Figure 3.2 below.

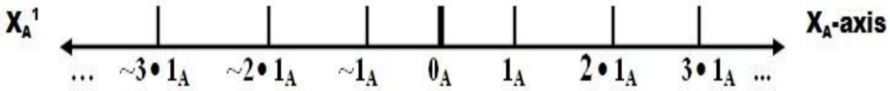

Figure 3.2 Object Linear Continuum Line of the complete region A with a collection of consecutive equi-spaced object points

The term 'equi-spaced' in the caption of Figure 3.2 is well understood in the sense of the corresponding metric (or norm) of the complete region A,

i.e. for any real integer r, $\rho(r\square\, 1_A, (r+1)\square\, 1_A)$ = constant (independent of r), in the complete region A by virtue of the beautiful property of 'Homogeneity' possessed by the metric ρ.
The following theorems follow directly from the above constructions:-

Theorem 3.1

(i) For every positive object x of the region space A, $x = \| x \| \square\, 1_A$

(ii) For every negative object x of the region space A, $x = \sim\| x \| \square\, 1_A$

Theorem 3.2

The unit element 1_A of the region space A is at the distance 1 (real integer) on the right side of the zero element 0_A on the object linear continuum line of A.

Theorem 3.3

The positive object x of the region space A is located at a distance $\| x \|$ from the zero element 0_A on the right side of 0_A on the object linear continuum line, and the negative object x is located at a distance $\| x \|$ from the zero element 0_A on the left side of 0_A,

Example 3.1

If we choose the region A to be the RR region which is a partitioned region with respect to the crisp order relation "Less Than or Equal To" denoted by the notation "≤", and if we choose $\|x\| = |x|$ in RR, where $\rho(x, y) = \|x-y\| = |x-y|$, then it can be observed that the X-axis of the region calculus is the classical X-axis popularly used by us in the Cartesian coordinate system in Geometry.

INTRODUCING "REGION GEOMETRY"

It it is observed that we can mathematically define infinite number of distinct 1-D complete regions in Mathematics. It is also mentioned that by a complete region, we shall always mean 1-D complete region. In the work (Biswas, 2026b) a new theory titled "Theory of Objects" is developed. We are now in a position to initiate a new kind of geometry on a complete region A. The "Theory of Objects" induces a new area which we call by 'Region Geometry' in a complete region. We begin the subject by introducing first of all a 2-D Region Geometry developed over an 1-D complete region. The new subject "Region Geometry" introduced here is in a baby stage today, but will surely take its volume soon in this era of AI in this century.
For developing the new geometry called by "Region Geometry", be it in a two dimensional region coordinate system or in an n-dimensional region coordinate system, at least one 1-D complete region A = (A,□,*,□) is required. The work in fact is sequel to the works (Biswas, 2026a, 2026b).
R is a trivial example of region. And it is explained that all the materials on the subject Geometry studied in school mathematics are based upon this region R. Subject to fulfilment of some conditions, a region A opens a new Geometry. Consequently, on the platform of the region R, the classical concepts of number line, coordinate plane, xaxis, y-axis, quadrants, x-coordinate of a point, y-coordinate of a point, location of a point on the coordinate plane, line segment AB, ray, line, etc are respectively termed as number line in the region R, coordinate plane in the region , x_R-axis, y_R-axis, quadrants, x_R-coordinate of a point, y_R-coordinate of a point, location of an object point on the R-coordinate plane, line segment AB in the R-coordinate plane, ray, line etc. The work in this work is done with respect to any region A, not just with respect

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to a particular region R. Every region A generates its own region geometry called by 'ARegion Geometry' subject to fulfilment of few conditions.

Consider the object linear continuum line and the X_A-axis corresponding to the complete region A. Consider a point x_A (a positive object) on the X_A-axis. Then for an infinitesimal small positive object Δx_A, the point (x_A □Δx_A) will be at a distance $\| \Delta x_A \|$ from the point x_A along the positive direction of X_A-axis and the point (x_A □Δx_A) will be at a distance $\| \Delta x_A \|$ from the point x_A along the negative direction X_A^1-axis; and in fact all the objects of the complete region A are well ordered in this sense, as explained earlier. In this section we incorporate "Y_A-axis" (imagine that a copy of X_A-axis is placed at right angle to the X_A-axis passing through the point 0_A, i.e. rotating through 90^0 anticlockwise about the point 0_A) and thus construct a region coordinate plane in the style of Cartesian coordinate system.

Every complete region has its own Region Geometry. We introduce first of all 2-D Region Geometry in a 1-D complete region A = (A,□,*,□). It is a system of geometry where the position of points on the plane is described using an ordered pair of objects, analogous to the case of Cartesian coordinate plane. We call this plane by 'Region Coordinate Plane'. A plane is a flat surface that goes on forever in both directions. If we were to place a point on the plane, region coordinate geometry gives us a way to describe exactly where it is by using two objects. Points are placed on the "region coordinate plane" as shown below in Figure 4.1. It has two scales: one running across the plane called the "X_A-axis" and another at right angles to it called the "Y_A-axis". Both these axes are thus object linear continuum lines corresponding to the complete region A.

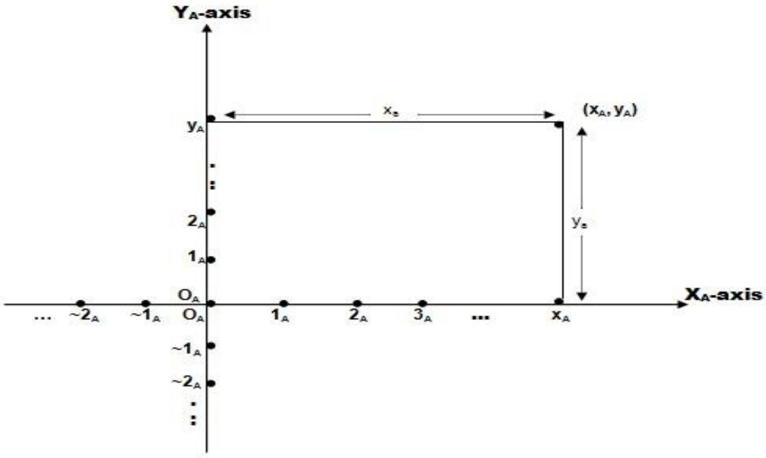

Figure 4.1 Object coordinates on region coordinate plane of the complete region A

The point where the two axes cross is called the origin denoted by O_A at which both x_A and y_A are 0_A. On the X_A-axis, as explained earlier that objects to the right of origin are positive objects and those to the left are negative objects of A. Similarly, on the Y_A-axis, objects above the origin are positive objects and those below the origin are negative objects of A.

A point's location on the region coordinate plane is given by two objects in the form of object coordinates (x_A, y_A), the first coordinate reveals where it is away from the Y_A-axis at parallel to the X_A-axis and the second coordinate reveals where it is away from the X_A-axis at parallel to the Y_A-axis (see Figure 4.1 above). There are four quadrants and sign convention rule is same as that of classical Cartesian coordinate

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geometry. If there is no confusion, we use the word X-axis instead of X_A-axis, Y-axis instead of $Y_A$axis in our literature here.

Consider the Region Geometry corresponding to the 1-D complete region A. Consider also the Region Geometry corresponding to the 1-D complete region RR. Thus there are two sets of Region Geometry we will consider now. Suppose that the region coordinate plane of A does also represent the region coordinate plane of RR taking same lines as two axes and taking the same location for origin (i.e. O_A and O_{RR} are co-incident points). Thus the X-axis, Y-axis, and the origin are common to both the region coordinate planes.

The results of the following proposition are straightforward.

Proposition 4.1

(1) If P be a point on the X-axis with the coordinates $(x_A, 0_A)$ on the region coordinate plane of A, then the coordinates of the same point P in the region coordinate plane of RR will be
  (i) $(\|x_A\|, 0)$ on X-axis, if x_A is a positive object,
  (ii) $(-\|x_A\|, 0)$ on X-axis, if x_A is a negative object.
  (i.e. the sign retaining rule will be followed, as the point P will remain in the same quadrant in both the region coordinate planes).

(2) If P be a point on the X-axis with coordinates $(x, 0)$ on the region coordinate plane of RR, then the coordinates of the same point P in the region coordinate plane of A will be $(x_A, 0_A)$ on the X-axis.
  (the sign retaining rule will be followed, as the point P will remain in the same quadrant in both the region coordinate planes).

(3) If P be a point on the Y-axis with coordinates $(0_A, y_A)$ on the region coordinate plane of A, then the coordinates of the same point P in the region coordinate plane of RR will be
  (i) $(0, \|y_A\|)$ on Y-axis, if y_A is a positive object,
  (ii) $(0, -\|y_A\|)$ on Y-axis, if y_A is a negative object.
  (the sign retaining rule will be followed, as the point P will remain in the same quadrant in both the region coordinate planes).

(4) If P be a point on the Y-axis with coordinates $(0, y)$ on the region coordinate plane of RR, then the coordinates of the same point P in the region coordinate plane of A will be $(0_A, y_A)$ on the Y-axis.
  (the sign retaining rule will be followed, as the point P will remain in the same quadrant in both the region coordinate planes).

(5) If P (x_A, y_A) be a point on the on the region coordinate plane of A, then the coordinates of the same point P in the region coordinate plane of RR will be one of the $(\square \|x_A\|, \square \|y_A\|)$ which is in compliance with the sign retaining rule, as the point P will remain in the same quadrant in both the region coordinate planes).

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(6) If P (x, y) be a point on the on the region coordinate plane of RR, then the coordinates of the same point P in the region coordinate plane of A will be (x_A, y_A). (the sign retaining rule will be followed, as the point P will remain in the same quadrant in both the region coordinate planes).

An object line is the unique line passing through two points on the Region Coordinate Plane of the region A. We now compute the slope of an object line for this region coordinate plane.

Slope of an object line passing through the two object points P(x_{1A},y_{1A}) and Q(x_{2A},y_{2A}) is the real number m_a given by (as shown in Figure 4.2) :

$$m_a = \tan\theta = \lambda . \frac{[illegible](y2A, y1A)}{[illegible](x2A, x1A)},$$ where λ is either +1 or -1 as per the usual sign rule followed in classical geometry.

Therefore, $m_a = \lambda . \left\| \frac{y^{2A}\ [illegible]\ y^{1A}}{x2A\ [illegible]\ x1A} \right\|$

$= \lambda . \left\| \frac{y2a\ [illegible]1A\ y1a\ [illegible]1A}{x2a\ [illegible]1A\ x1a\ [illegible]1A} \right\|$

$= \lambda . \left\| \frac{(y^{2a}\ [illegible]y1a)\ 1[illegible]\ A}{(x2a\ [illegible]x1a)\ 1[illegible]\ A} \right\|$

$= \lambda . \left| \frac{y^{2a}\ [illegible]y1a\ .\ 1A}{x2a\ [illegible]x1a\ .\ 1A} \right|$

$= \lambda . \left| \frac{y^{2a}\ [illegible]y^{1a}}{x2a\ [illegible]x1a} \right|$

$= \lambda . \left| \frac{y2.1a\ [illegible]y1.1a}{x2.1a\ [illegible]x1.1a} \right|$, where y_{2a} means $(y_2)_a$ which is equal to $y_2.1_a$

$= \lambda . \left| \frac{y^2\ [illegible]y1}{x2\ [illegible]\ 1} \right|$

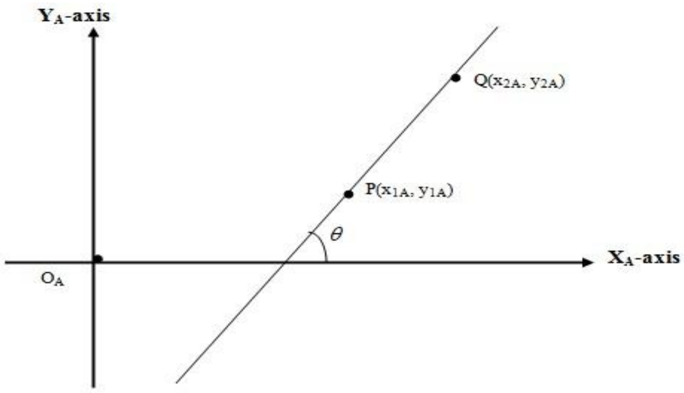

Figure 4.2 Slope of an objects line

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Thus the slope is neither dependent upon the concerned region nor upon the metric of the region.
Distance between two object points on a region coordinate plane can be defined in various ways like in classical geometry. However, we follow the style of Euclidian distance in Region Geometry here.
Consider the X_AY_A region coordinate plane corresponding to the complete region A. Let $P(x_{1A},y_{1A})$ and $Q(x_{2A},y_{2A})$ be two points on this region plane (see Figure 3.3). Distance PQ between these two points is the positive real number d, where

d = □((□ y y2A, 1A))² □((□x x2A, 1A))²□ ½ .

It can be computed that this distance is an absolute distance in the sense that neither it is dependent upon the concerned region nor upon the metric of the region.
We see that,

d2 = ‖y2A y1A ‖2 □ ‖x2A x1A‖2 = y2a □1A y1a □1A 2 □ x2a □1A x1a □1A 2 ‖ ‖ ‖2 ‖ ‖2

= (y2a □y1a) 1□ A □ (x2a □x1a) 1□ A 2 2 (x2a □x1a) .12 2a

= (y2a □y1a) .1a □ 2 (x2a □x1a) }²

= {(y2a □y1a) □ 2 (x2 □x1) }.²

= {(y2 □y1) □

This implies that d is neither dependent upon the concerned region nor upon the metric of the region.

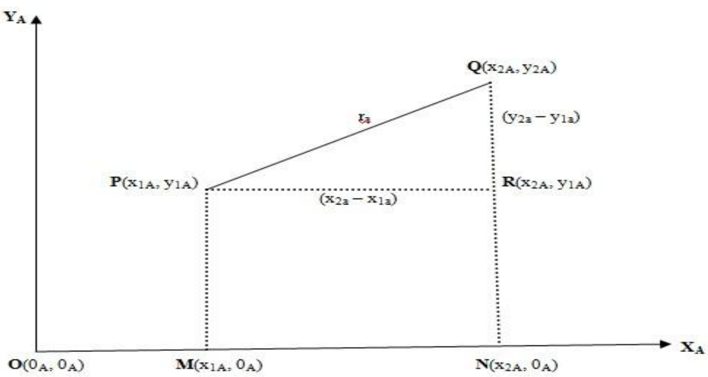

Figure 4.3 Distance between two object points

The Pythagoras Theorem is thus valid irrespective of the concerned region or the metric of the region.

Proposition 4.2

Pythagoras Theorem is valid in every Region Geometry, whatever be the corresponding complete region A.

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OBJECT LINE

Consider the X_AY_A region coordinate plane corresponding to the complete region A (see Figure 5.1). The general equation of an object line whose slope is m_a is
$y_A = m_a \square x_A \square c_A$.

Equation of an object line having slope m_a and passing through the object point $Q(x_{1A},y_{1A})$ is

$$(y_A \sim y_{1A}) = m_a \square (x_A \sim x_{1A}).$$

Equation of an object line passing through the two object points $P(x_{1A},y_{1A})$ and $Q(x_{2A},y_{2A})$ is

$$(y_A \sim y_{1A}) = m_a \square (x_A \sim x_{1A}), \quad \text{where } m_a = \frac{y_{2a} \square y_{1a}}{x_{2a} \square x_{1a}}.$$

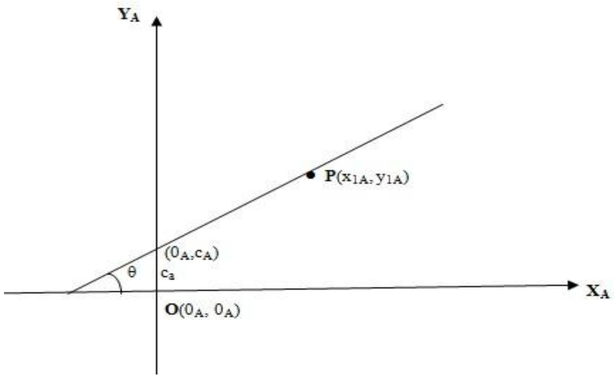

Figure 5.1 An object line having positive intercept of length c_a on Y_A axis.

If an object line MN has the intercepts of lengths p_a and q_a on the X-axes and Y-axes respectively at the points $(p_A, 0_A)$ and $(0_A, q_A)$, then the equation of this object line MN will be (as shown in Figure 5.2):

$$\frac{x_A}{p_a} \square \frac{y_A}{q_a} \square 1_A$$

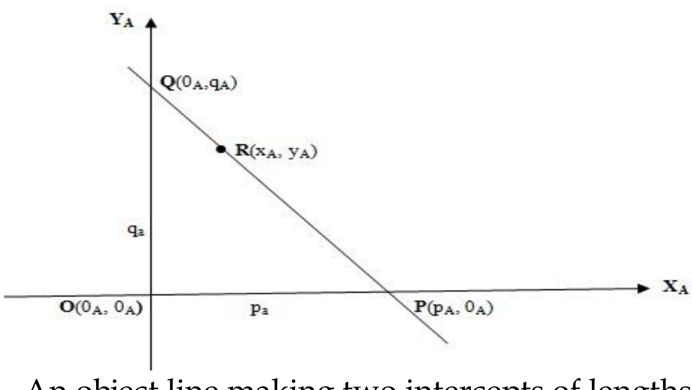

Figure 5.2 An object line making two intercepts of lengths p_a and q_a.

In the above equations, the variables take values which are objects of the region A. The classical geometry taught at school level is a particular instance of Region Geometry based upon a particular region which is R.

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OBJECT CIRCLE

Consider the X_AY_A region coordinate plane corresponding to the complete region A. Then the equation of an Object Circle (see Figure 6.1) with centre at $(0_A, 0_A)$ and radius r_a (>0) is given by

$$((\square x_A, 0\ ))_A\ ^2\ \square((\square y_A, 0\ ))_A\ ^2\ \square r_a^2,$$

which can be written as

$$\| x_A \|^2\ \square \| y_A \|^2\ \square r_a^2.$$

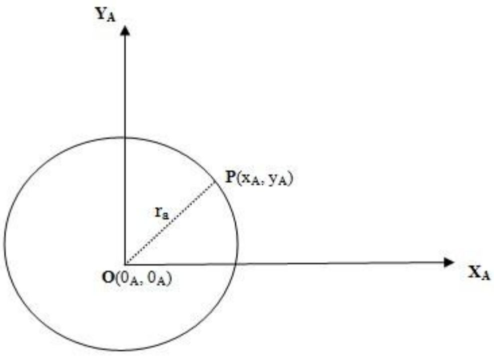

Figure 6.1 Object circle with centre at origin

And the equation of the Object Circle (see Figure 6.2) with centre at (α_A, β_A) and radius r_a (>0) is given by

$$((\square\ \square\ \square\ \square x_A,\ _A))^2\ \square((y_A,\ _A))^2\ \square r_a^2,$$

which can be written as

$$\| x_A\ \square_A 2 \|\square\ y_A\ \square_A 2\ \square \| r_a 2.$$

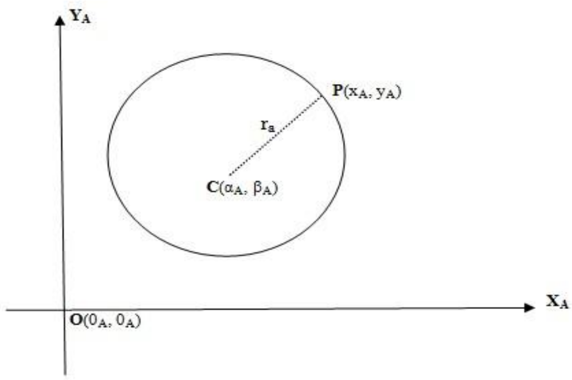

Fig. 6.2 Object circle with centre at the object point **C**(α_A, β_A)

Further study on Region Geometry can be easily done analogous to the literature of classical geometry.

CONCLUSION

In today's AI era of this century, most of the complex data being analysed by the researchers are unstructured or semi-structured data, not the data of R or R^n. Consequently, the collected data are not always from R or R^n, but various types of objects. Todays AI-analysts need more support from Pure Mathematics to deal with unstructured or semi-structured data in their everyday analysis and research activities.

The "Region Geometry" is initiated to cater to today's requirements in the AI-era in this century. "Region Geometry" corresponding to the particular region R generates the classical subject "Geometry". Today the newly born subject :Region Geometry" is in its baby stage, but it is expected that in due time the "Region Geometry" will get fast growth to play good roles in Mathematics, Statistics, Engineering branches, Medical Science, Data Science, Artificial Intelligence and Superintelligence in the era of AI in this century.

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